THE MATHEMATICAL, PHILOSOPHICAL AND EDUCATIONAL IMPACTS OF TIME DILATION AND LENGTH CONTRACTION

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Abstract

Mathematically speaking, it is justifiable to state that 1=1. If 1=1, then 1=1xA is also correct if and only if A=+1. But can the same be said of 0=0? It is important to bear in mind that while 1 is a natural number, zero is not! There are difficulties associated with certain mathematical operations involving zero, such as dividing zero by zero, for they result in indeterminate values. Hence, unlike 1=1xA, the premise 0=0xA would be correct for any value of A. Thus, any theory founded upon 0=0xA, or any premise that permits more than one mathematical solution must be considered unsound for it would be liable to costly errors in its power to interpret the physical world accurately. In the course of the present work we will show that Standard Special Relativity is apparently tainted with a mathematical premise that is more inclined towards accepting 0=0×A. The mathematical, philosophical and pedagogical impacts of such a premise will be discussed briefly in respect to the Lorentz factor, time dilation, length contraction and Immanuel Kant’s concept of synthetic a priori.

Keywords: Lorentz factor, Time dilation, Length contraction, synthetic a priori

1 INTRODUCTION

Although written in 1905, Einstein's Special Theory of Relativity, published in his Zur Elektrodynamik bewegter Körper (On the Electrodynamics of Moving Bodies) in 1905, is still considered the standard theoretical approach in relativistic physics to measure and explain the relationship between space (and/or length in the direction of motion) and time within inertial frames of reference. Time and length are fundamental concepts in physics. For inertial observers in relative motion, these fundamental concepts can be either absolute or relative and they can have the same or different magnitudes, depending on ‘Lorentzian’, ‘Minkowskian’, ‘Machian’, ‘Leibnizian’, ‘Maxwellian’, ‘Galilean’, ‘Newtonian’ and ‘Aristotelian’ spacetimes. Unlike classical physics, the physics of special relativity rejects the concept of absolute time and length. Instead, special relativity theory considers that the same time and length spans have different magnitudes for different inertial observers in relative motion. We will assess in the present work some of the implications of these relativistic concepts according to Albert Einstein’s theory of special relativity, and show that using premise 0=0 to build a theory of special relativity would lead to an incorrect conclusion on length contraction. However, the same premise surprisingly leads to a conclusion on time dilation. This paradox can be explained mathematically as one of the misleading implications of the premise 0=0. For example, the conclusion 2=3 can result if we consider 2 × 0 = 3 × 0; then, by cancelling zero on both sides of the equation would result in a '2=3' fallacy.

2 The MATHEMATICAL PREMISE OF STANDARD SPECIAL RELATIVITY

Science generally relies upon mathematical justifications to support premises about the physical world. In the absence of direct experimental evidence, the mathematics used to support a theory needs to be even more rigorous. Albert Einstein's theory of special relativity was presented to the world in 1905, and for more than a century the verdict of mathematicians has accepted its mathematical premises as valid. At first, it was judged on the basis of its mathematical formulations (supported by thought experiments). In the course of time Einstein's equations and thought experiments seemed to receive experimental confirmation. As long as the theory continues to receive experimental support, even to the present day, then why question the mathematics? But things might not be as they seem!

As far as we are aware, there has never been a peer reviewed paper that aimed to evaluate a main mathematical premise from which the equations of Einstein's special relativity can surface i.e. 0=0. We believe that this starting point needs to be assessed,

Einstein's standard model of special relativity is inclined to accept the premise 0=0, which is undoubtedly a mathematically justifiable premise. Our alternative special relativity (to be published) begins with another mathematically justifiable premise, 1=1. Choosing to establish equations for special relativity on either 0=0 or 1=1 will have different implications. The question then becomes whether the implications of the two approaches are mathematically consistent.

Let us first consider the mathematical veracity of the premise 1=1. If 1=1 then 1=1×A is also correct *if and only if* A= +1. This is due to the possibility of A≠1; for example, if A=0 then 1=0 and if A= -1 then 1= -1, respectively! It is mathematically unacceptable for 1≠1. As only one mathematical value is accepted by 'A', namely +1, 1=1×A is a sound mathematical premise. Any proposition that permits or gives the possibility for more than one mathematical solution would be considered an unsound premise in mathematics.

Evidently, the premise 1=1 produces clear mathematical results and has to be accepted as a sound mathematical premise upon which a theory of special relativity can be established. Can the same be said of the premise 0=0?

It is important to bear in mind that while 1 is a *natural number*, zero is not. Mathematically speaking, there are simply unavoidable difficulties associated with certain operations involving zero. These include dividing zero by zero (0/0) and zero to the power of zero (0^0), which both result in indeterminate values. Hence, unlike 1=1×A, the premise 0=0×A would be correct for any value of A. Thus, any theory founded upon 0=0×A, or any other premise that permits more than one mathematical solution must be considered unsound for it would be liable to costly errors in its power to interpret the physical world accurately.

Einstein's special relativity is apparently tainted with a mathematical premise that is more inclined towards accepting 0=0×A! We can demonstrate how this premise (0=0) is inherent in the standard model of special relativity by considering the expression $x=ct$ or $x-ct=0$ where x is the distance travelled by the photon in t seconds of time, and c is the speed of light. Such a premise can lead to peculiar mathematical implications such as 0/0. Although the premise '0=0' is not explicitly declared by the mathematics of the standard approach to special relativity, in respect to issues related to the light-cone, it is nevertheless one of the overlooked premises in the theory:

$$0 = 0 \qquad \text{(mathematical premise)}$$

$$0 = A\times 0 \qquad \text{(where A is any quantity)}$$

$$0 = B\times 0 \qquad \text{(where B is any quantity).}$$

$x^2 - c^2t^2 = (x-ct)(x+ct) = x'^2 - c^2t'^2 = (x'-ct')(x'+ct') = 0$ (a mathematical premise in the standard approach to special relativity) where $x - ct = 0$ (in the system s) and $x'-ct' = 0$ (in the system s') (Einstein A., 1907).

Standard special relativity expresses the kinematics of light pulses propagating in parallel in system s and s' as:

$x' - ct' = A\,(x - ct)$. This, we believe, is indisputably "$0 = A\times 0$" in disguise.

$x' + ct' = B\,(x + ct)$. This, we believe, is indisputably "$0 = B\times 0$" in disguise.

Any theoretical approach founded upon such a mathematical stance is bound to be fraught with problems as we shall show.

Solving these last two equations for x' and ct' yields:

$$x' = x(A + B)/2 - ct(A - B)/2$$

$$ct' = ct(A + B)/2 - x(A - B)/2$$

or

$$x' = xR_1 - ctR_2 \quad \text{where } R_1=(A + B)/2, \text{ and } R_2=(A - B)/2.$$

$$ct' = ctR_1 - xR_2$$

Here, for $x' = 0$, we get $x = ctR_2/R_1$ or $x/t = v = cR_2/R_1$. Form the standard approach to special relativity point of view, v here is the velocity with which the origin of s' is moving relative to s. Notice now that $x=vt$ (or put another way, $x-vt=0$). This is a fundamental problem that is at odds with the initial premise of the standard approach to special relativity: $x- ct=0$! Notice if we, under this *oddity*, use $v=cR_2/R_1$ in the relation $x'=xR_1 - ctR_2$ we arrive at $x'=R_1(x- vt)$. Now comparing $x'=R_1(x- vt)$ with Lorentz equation for length-contraction: $x' =(x- vt) / [1- (v/c)^2]^{½}$, we find that R_1=Lorentz factor = $\gamma= 1/ [1- (v/c)^2]^{½}$!

If $x=ct$ and $x=vt$, then, simply put, this scenario tells us that $v=c$! If $v=c$ then there can be no way of developing a theory of special relativity because a moving system (s') will be moving at the speed of light. For relativity to be a valid notion, $v\neq c$ or according to the standard approach to special relativity the velocity of any mass-ive object must be less than the speed of light ($v<c$).

In the above approach, '*0=0*' is an inherent premise that comes at a high cost. It makes the approach impotent. It only takes a few mathematical steps, from the above, to obtain the Lorentz transformation equations:

$$x' = (x- vt) / [1- (v/c)^2]^{½}$$

$$t' =[t - (vx) /c^2] / [1- (v/c)^2]^{½}$$

(Here if $x' = 0$ corresponds to $x – vt = 0$, then $0= \gamma\times 0$ or $\gamma= 1/ [1- (v/c)^2]^{½} = 0/0$!? Dividing zero by zero is considered mathematically indeterminate and could be the reason why, as shall be argued, the standard approach to special relativity lacks consistency in respect to length contraction?).

The first equation: $x' =(x- vt) / [1- (v/c)^2]^{½}$ does not, however, lead to length-contraction, as has been incorrectly assumed. The claim that the equation provides a proof for length-contraction simply does not withstand mathematical scrutiny. If x/t is considered greater than v (or alternatively, $x > vt$) then a case for length-contraction could be made: $x' = X / (1- (v/c)^2)^{½}$ where $X=x-vt$ is the Galilean transformation: $X=ct-vt$, if $x=ct$. Unfortunately, such a distinction cannot be made between the two variables (x and vt) as v here is a quantity by which x/t is expressed: what is v if not x/t ?

It should be noted that in the equation $x' =X/ (1- (v/c)^2)^{½}$ only X will contract, whereas x' elongates (quite the opposite of what the standard approach to special relativity demands)! We need to be clear about what X represents: so what is X? X should not be confused with x, nor with x'. As such, it should be clear that X is definitely not the x found in standard special relativity's equation for length (and/or space) contraction:

$$x' =x\,(1- (v/c)^2)^{½}$$

Even if we assume for the sake of argument that this, now controversial, length-contraction equation $x' =X/(1-(v/c)^2)^{½}$ is correct, there should be absolutely no doubt that it cannot be claimed as a proof for length contraction and, therefore, should be questioned as a valid template for the length-contraction equation in standard special relativity.

Surprisingly, in regard to the second equation for time dilation: $t'=[t- (vx)/c^2] / [1- (v/c)^2]^{½}$, no similar controversial implications exist. The equation does seem to be mathematically sound, thus leading to the time-dilation equation in standard special relativity:

$t' =t(1- (v/c)^2)^{½}$ but only on the condition that $v=x/t$. This can be demonstrated:

$$t' = [t - (vx)/c^2] / [1- (v/c)^2]^{½}$$

$$t' = [t - (x^2)/tc^2] / [1- (v/c)^2]^{½} \quad \text{(substituting } v=x/t \text{ in the term: } t - (vx)\text{)}$$

$$t' = [t - (x^2t)/t^2c^2] / [1- (v/c)^2]^{½} \quad \text{(multiplying the term } x^2/tc^2 \text{ by } t/t\text{)}$$

$$t' = [t - (v^2t)/c^2] / [1- (v/c)^2]^{½} \quad \text{(substituting } x^2/t^2= v^2\text{)}$$

$$t' = t\,[1 - (v^2/c^2)] / [1- (v/c)^2]^{½} \quad \text{(} t \text{ being the common factor).}$$

This is Einstein's well known time dilation equation:

$$t' = t\,[1- (v^2/c^2)]^{½}.$$

It should once again be emphasized that compatibility between Lorentz and Einstein here can only occur if, and only if, $v=x/t$.

Interestingly, while $v=x/t$ jeopardizes the first equation it validates the second equation. For a simple expression to be sound in one instant but not in one directly related to it is surely a worry for the mathematical reliability of the standard approach to special relativity.

3 PHILOSOPHICAL IMPLICATIONS

Kant categorized 'universal' knowledge that did not rely on the senses as *a priori* while referring to knowledge dependent on the senses as empirical. Kant considered Euclidean Geometry, for instance, to be *a priori* rather than empirical because the concept of the interior angles of a triangle being 180° is not implicitly contained in the concept of a triangle (a three sided shape). In fact, space and time are categorised as *synthetic a priori* postulations by Kant. Einstein's relativity was a serious challenge to this categorisation in Kant's philosophy; as it was to Kant's distinction between analytic propositions (true by virtue of their meaning) and synthetic propositions (true by how their meaning relates to the world). Surely, one theorist's definition could be another theorist's synthetic, empirical claim. Knowledge of the universe requires evidence from the universe rather than categorising propositions as *a priori* – synthetically or analytically. Einstein supported the empiricist tenet that there are no *synthetic a priori* truths (Howard D. A., 2005).

Einstein's relativity, which is the cornerstone of modern physics, does not rely on Euclidean Geometry so neo-Kantians might argue as Euclidian Geometry could be still understood as *synthetic a priori* but both Euclidean Geometry and Einstein's model view space-time as an empirical postulation. General relativity (where the geometry of space-time is curved) and Special Relativity (where geometry can be said to be flat) can only work if space-time is empirical. On the other hand, we should not assume that Einstein can neatly be categorised as an ardent empiricist. He would have agreed with Kant's view that the mind is not simply a blank slate upon which experience freely writes – cognition must involve some structuring by the knower. Einstein was dismayed by the extreme empiricist anti-metaphysical doctrine held by the Vienna Circle that dismissed as metaphysical any element of theory whose connection to experience could not be demonstrated clearly enough. (Howard, D. A., 2005).

Einstein's dismay was hardly surprising when we realise that he relied upon mathematics and thought experiments to formulate his theory rather than actual experimental evidence. Einstein fully expected that experimental evidence was needed to finally confirm his theory, and it did come, but interestingly enough his equations and thought experiments were so convincing that he gained a large following even before his theory received experimental confirmation. His supporters were confident that this would soon come. In the mean time they were convinced that his approach to relativity was ultimately empirical.

The question now arises: If Einstein had finally put to rest Immanuel Kant's famous assertion that Euclidean Geometry was a *synthetic a priori truth* rather than an empirical one then why are we proposing an alternative new theory of special relativity? Could it be the case that Einstein's special relativity, and even mathematics as a whole, is *synthetic a priori*? We know that special relativity uses the concept of reference frames –an observational perspective in space which is not undergoing any change in motion (acceleration), from which a position can be measured along 3 spatial axes. Might this mean that, in some way, there can be alternative theories of relativity depending on one's reference frame?

But this would indeed relegate special relativity to simply being a product of the human mind – perhaps a social, cultural or psychological construction. No! Our special relativity supports Einstein's account that space-time is empirical. It should not be overlooked that Einstein's special relativity is experimentally *well-confirmed* and this means that the theory has a foundation in the empirically testable world. However, using the phrase "well-confirmed" rather than simply "confirmed" indicates that there are still conundrums in some of the conclusions of special relativity that have not been adequately addressed by its supporters.

These include the assumption that moving at speeds approaching the speed of light will cause an object to experience length contraction (to zero), time dilation (also to zero) while simultaneously experiencing an increase in mass to infinity. How is it possible to possess length (in the direction of the motion) equalling zero yet also have a mass equal to infinity? Some of these counter intuitive implications in standard special relativity are due to Einstein’s insistence on employing the Lorentz factor, as an essential element in his formulations. The Lorentz factor sets a limit to speed in the universe – which might not be warranted. Some of the counter intuitive implications of standard special relativity can simply be avoided by rejecting the idea that the speed of light is a censor on speed.

Also, as already argued, the starting premise 0=0, explains many of the peculiarities in the standard approach. Furthermore, by introducing the concept of space-time, physicists are no longer free to distinguish between an independent state called space nor one called time. Einstein forced an amalgamation of space and time so that while space-time became an important concept within the model of special relativity to understand the universe, the concepts of space and time nevertheless became merged into something rather unresolved.

Supporters of Einstein’s special relativity have tended to dismiss critics who point out counter intuitive implications as possessing insufficient understanding of the basics of special relativity, especially how the Lorentz transformation can bring about length contraction and time dilation.

While it might be admitted that standard special relativity can be considered a mathematical truth its tendency to create counter intuitive scenarios raises questions when considering its validity as an empirical truth. In contrast, alternative special relativity theories need to escape the extreme counter intuitive implication of standard special relativity – if they hope to rival or contribute anything more than Einstein accomplished in his 1905 paper. Avoiding the counter intuitive implications of Einstein’s special relativity, along with the standard model’s insistence on the vague concept of space-time, would have enormous pedagogical implications. Students would be able to study special relativity with a clearer understanding of the meaning of space and also of time. Educators would have less difficulty teaching special relativity to students, and students would have a more positive experience, if an alternative special relativity can actually emerge that does not constantly expect us to believe in counter intuitive outcomes.

4 PEDGOGICAL IMPLICATIONS

Educationalists who have attempted to introduce Einstein’s relativity to high school students and undergraduates with limited knowledge of mathematics and science have realised that students simply find the theory and implications of Einstein’s mathematics to be counter intuitive. Yet, the same educationalists would deny that either the mathematics or descriptions of Einstein’s relativity are in any way paradoxical. This is one of the problems then, in teaching special relativity, the teacher is so convinced by the theory that they themselves see no logical contradictions or paradoxes.

We would actually endorse the view of these educationalists. The mathematics of the standard approach to special relativity is neither paradoxical nor poses mathematical contradictions (though implicitly inclined to accept premises such as 0=0). The problem is that when students notice the extent to which the implications of the mathematics of special relativity are counter intuitive they, especially students who are unable to master the mathematical principles, do conclude that as the implications are counter intuitive then special relativity is indeed paradoxical – countering logic, senses experience and our general understanding of the physical world.

Interestingly enough, educationalists who have advocated teaching Einstein’s theories of relativity to non-scientist undergraduates, non-mathematicians and to high school students, begin by claiming that there are no “paradoxes” in the standard theory while simultaneously having to admit that the theory has “peculiarities”. It is tempting to wonder whether the word “peculiarity” is simply a synonym for “paradox”. For example, Mermin has no qualms using both words on a single page of seminal work on the issue of pedagogy and relativity:

> “The most important thing I learned in teaching relativity to many generations of Cornell undergraduates, none of them science majors, is that one must begin teaching them the techniques of changing frames of references by applying that technique to some entirely commonplace, highly intuitive example. There are many such ways to develop these skills, and they enable one to learn much that is not at all obvious, though never *paradoxical* … [O]ne should emphasize as early as possible that although objects moving at the speed of light famously behave in some *very strange*

ways, the behaviour of objects moving at speeds comparable to the speed of light can be just as *peculiar*. The *peculiarity* of motion at the speed of light can be just a special case of a more general *peculiarity* of all motion, which becomes prominent only at extremely high speeds." (Mermin, 2005, our italics).

Teachers engaged in teaching the standard model of relativity have to persuade students that "things" simply behave in *peculiar* and *very strange* ways when travelling at extreme speeds – but these are never *paradoxical*. Yet, should we insist that students subscribe to some of the more counter intuitive claims of the standard theory? e.g. as an object travels at speeds approaching the speed of light it will experience such drastic changes as time reducing to zero, (object's) length (in the direction of the motion) diminishing to zero, but (object's) mass will at the same time soars to infinity?

The Einstein-First movement, enthusiastic about Einstein's theories of Relativity, have proposed that as Einstein's relativity is unanimously accepted as the best scientific understanding of the universe, students should forego learning Newton's physics until they learn the fundamental concepts of Einstein's physics. Their reasoning is based on the assumption that if young and malleable minds learn about relativity first they would not face any contradictions in their prior knowledge as is the case for most students who encounter Newton first and then Einstein. This perspective assumes that if students grasped the concepts of relativity first, they would easily be able to learn that, according to Newton and for most of the time on Earth, we can treat time as being absolute, space as being flat, and gravity as a force field coming out of the Earth. This is because our perception and experience of the world coincides more with Newton's physics than with Einstein's. But such an approach is again counter intuitive as Einstein's mathematics is more challenging so why expose students to that which is more difficult rather than scaffolding their learning in stages from Newton first and then to Einstein?

However, the Einstein-first approach, as it is called, argues that physics can be taught from an observational point of view, without recourse to complex abstract mathematics (Kaur et al, 2017). Although Einstein's mathematics is no more challenging than those of Newton we do not believe that the main difficulty for students learning about relativity derives from complex mathematical equations. The main obstacle for students is, in our opinion, the fact that many conclusions resulting from relativity are simply counter-intuitive. There can be little doubt that younger students, especially, would be convinced by and be more open to the observable conclusions of Newton than the more counter-intuitive ones provided by relativity. The fact that we are now proposing an alternative perspective to Einstein's Special Relativity would present further challenges to younger minds attempting to simply understand the subject of physics.

5 CONCLUSIONS

In co-local events (such as s and s' moving simultaneously with respect to each other), light speed constancy demands $x=ct$ ($x^2=c^2t^2$ or $x^2 - c^2t^2 = 0$) in s-frame, and $x'=ct'$($x'^2=c^2t'^2$ or $x'^2 - c^2t'^2 = 0$) in s'-frame. If so, then $x^2 - c^2t^2 =0= x'^2 - c^2t'^2$. The arbitrary application of this premise under the conditions: $x=0$ for $t\neq0$ in the s-frame, and $x'=0$ for $t'\neq0$ in the s'-frame; or alternatively $t=0$ for $x\neq0$ in s-frame, and $t'=0$ for $x'\neq0$ in the s'-frame, is shown here to result in different conclusions for length contradiction and time dilation according to the standard approach to special relativity as theorized by Einstein. Pedagogically speaking, this conclusion would present further challenges for educators and students alike. Attempting to teach and learn standard special relativity theory will continue to be plagued with counter intuitive scenarios that would inevitably diminish the possibility for both educators and students to fully grasp special relativity.

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